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WITH DISCONTINUOUS INPUTS**

by

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# TIME-DEPENDENT LINEAR DAE'S WITH DISCONTINUOUS INPUTS<sup>1</sup>

BY

PATRICK J. RABIER AND WERNER C. RHEINBOLDT<sup>2</sup>

**ABSTRACT.** Existence and uniqueness results are proved for initial value problems associated with linear, time-varying, differential-algebraic equations. The right-hand sides are chosen in a space of distributions allowing for solutions exhibiting discontinuities as well as "impulses". This approach also provides a satisfactory answer to the problem of "inconsistent initial conditions" of crucial importance for the physical applications. Furthermore, our theoretical results yield an efficient numerical procedure for the calculation of the jump and impulse of a solution at a point of discontinuity. Numerical examples are given.

## 1. Introduction.

In this paper, we prove existence and uniqueness results for initial value problems associated with differential-algebraic equations (DAE's) in  $\mathbb{R}^n$

$$(1.1) \quad A\dot{x} + Bx = b,$$

where  $A, B$  are smooth time-varying linear operators, and  $b$  belongs to a class of distributions with values in  $\mathbb{R}^n$  containing the functions that are smooth in  $(-\infty, 0]$  and  $[0, \infty)$  and have a discontinuity at the origin. Such discontinuities on the right side occur frequently in physical problems modelled by DAE's. For instance, in electrical network problems a discontinuity of  $b$  may correspond to the operation of a switch at a given time.

The existence and uniqueness theory for problems of the form (1.1) with smooth  $b$  (and consistent initial conditions) is now well understood, see, e.g., [C87], [KuM92], [RR93a] and

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the references given there. But elementary examples show that the setting of distributions is indispensable for handling discontinuous right-hand sides. For example, if  $A$  is constant with  $A^2 = 0$ ,  $B = I$ , and  $b = b_0 H$  where  $b_0 \in \mathbb{R}^n$  and  $H$  is the Heaviside function, then the solution of (1.1) (in this case unique: no initial condition needs to be or should be prescribed) is  $x = b_0 H - A b_0 \delta$  and hence involves the Dirac delta distribution.

When  $A$  and  $B$  are constant and  $b$  exhibits jumps, Laplace transform methods are available to find the solutions of (1.1), but the problem appears to remain open for time dependent coefficients  $A, B$ . This is the case considered in this paper. Our results depend essentially upon our recent work [RR93a] on a reduction procedure that transforms the distribution solutions of (1.1) into the distribution solutions of an explicit ODE.

The "consistency" of initial conditions represents another topic of considerable theoretical and practical importance in the study of DAE's. As is well known, even for smooth  $b$ , (1.1) will not have a solution starting at arbitrary points  $x_0 \in \mathbb{R}^n$ . Rather, existence of a solution in the classical sense requires that  $x_0$  satisfies certain constraints called the consistency conditions. On the other hand, suppose that the physical process modelled by (1.1) starts at time  $t = 0$ , and that for  $t < 0$  the state variable  $x(t)$  has evolved in a way totally unrelated with (1.1). If  $\lim_{t \rightarrow 0^-} x(t) = x_0$  exists this  $x_0$  represents a natural data value for the initial condition at  $t = 0$ . But, since  $x_0$  has no reason to be consistent with (1.1) at  $t = 0$ , the mathematical theory only provides that (1.1) has no solution for this choice of initial condition, which is, of course, a physically unacceptable statement.

It turns out that the consistency question is closely related to the problems addressed here. By viewing this question as that of extending a known state  $x(t)$  for  $t < 0$  to a solution of (1.1) for  $t > 0$  via a solution of (1.1) in  $(\mathcal{D}'(\mathbb{R}))^n$ , we show that the ambiguity can be resolved: From  $x_0 = \lim_{t \rightarrow 0^-} x(t)$  we find that a unique, computable jump to a consistent value occurs at  $t = 0$ . Furthermore, for problems with index  $\nu \geq 2$ , the sudden transition between  $x_0$  and the consistent initial value may also create a (computable) impulse; that is, a linear combination of  $\delta$  and its derivatives. Further evidence that our solution is the correct one is provided by showing that it is the limit of the classical solutions of the problems obtained by smoothing out the right-hand side near  $t = 0$ . These results

complement in various ways those already obtained for problems with constant coefficients in [VLK81], [Co82], or [G93].

Section 2 gives a brief review of the reduction procedure developed in [RR93a]. Initial value problems for (1.1) are then considered in Section 3 for right-hand sides in a class of distributions which is a close relative of the class  $C_{imp}^n$  of "impulsive-smooth" distributions introduced in [HS83]. The application of these results to the problem of inconsistent initial conditions is discussed in Section 4, and some straightforward generalizations are presented in Section 5. Finally, Section 6 presents some computational results that illustrate the resulting algorithms.

## 2. Reduction procedure for linear DAE's.

Let  $A, B \in C^\infty(\mathbb{R}; \mathcal{L}(\mathbb{R}^n))$  and set  $A_0 = A$ ,  $B_0 = B$ . The reduction procedure developed in [RR93a] generates, under appropriate conditions, a new pair  $(A_{j+1}, B_{j+1})$  from the pair of coefficient functions  $(A_j, B_j)$ ,  $j \geq 0$ . More precisely, set  $r_{-1} = n$  and assume that  $A_j, B_j \in C^\infty(\mathbb{R}; \mathcal{L}(\mathbb{R}^{r_{j-1}}))$  for some integer  $0 \leq r_{j-1} \leq n$ . Moreover, suppose that

$$(2.1) \quad \text{rank } A_j(t) = r_j, \quad \forall t \in \mathbb{R},$$

where  $0 \leq r_j \leq r_{j-1}$  is a fixed integer, and that

$$(2.2) \quad \text{rank } A_j(t) \oplus B_j(t) = r_{j-1}, \quad \forall t \in \mathbb{R},$$

where  $A_j(t) \oplus B_j(t) \in \mathcal{L}(\mathbb{R}^{r_{j-1}} \times \mathbb{R}^{r_{j-1}}, \mathbb{R}^{r_{j-1}})$  is defined by  $A_j(t) \oplus B_j(t)(u, v) = A_j(t)u + B_j(t)v$ .

Under the conditions (2.1) and (2.2), it is shown in [RR93a] that the following mappings exist:

- (i)  $P_j \in C^\infty(\mathbb{R}; \mathcal{L}(\mathbb{R}^{r_{j-1}}))$  such that  $P_j(t)$  is a projection onto  $\text{rge } A_j(t)$ ,  $\forall t \in \mathbb{R}$ .
- (ii)  $C_j \in C^\infty(\mathbb{R}; \mathcal{L}(\mathbb{R}^{r_j}, \mathbb{R}^{r_{j-1}}))$  such that  $C_j(t) \in GL(\mathbb{R}^{r_j}, \ker Q_j(t)B_j(t))$ ,  $\forall t \in \mathbb{R}$ , where  $Q_j = I - P_j$ .

(iii)  $D_j \in C^\infty(\mathbf{R}; \mathcal{L}(\mathbf{R}^{r_{j-1}}, \mathbf{R}^{r_j}))$  such that  $D_j(t) \in GL(\text{rge } A_j(t), \mathbf{R}^{r_j})$ ,  $\forall t \in \mathbf{R}$ .

With  $C_j$  and  $D_j$  as in (ii) and (iii) above, we define

$$(2.3) \quad A_{j+1} = D_j A_j C_j, \quad B_{j+1} = D_j (B_j C_j + A_j \dot{C}_j),$$

so that  $A_{j+1}, B_{j+1} \in C^\infty(\mathbf{R}; \mathcal{L}(\mathbf{R}^{r_j}))$ .

If (2.1) and (2.2) hold for every index  $j \geq 0$ , the procedure can be continued indefinitely. At the same time, since the sequence  $r_j$  is non-increasing, there is a smallest integer  $\nu \geq 0$  such that  $r_\nu = r_{\nu-1}$ . By (2.1), we then have  $A_\nu(t) \in GL(\mathbf{R}^{r_{\nu-1}})$ ,  $\forall t \in \mathbf{R}$ , and further reductions produce pairs  $(A_j, B_j)$ , equivalent to the pair  $(A_\nu, B_\nu)$  in a sense defined in [RR93a]. The integer  $\nu \geq 0$  is called the index of the pair  $(A, B)$ , and it can be shown that  $\nu$  is independent of the specific choices of  $P_j$ ,  $C_j$  and  $D_j$ ,  $0 \leq j \leq \nu - 1$  made during the process.

**Remark 2.1.** For constant  $A$  and  $B$  it can be shown that  $(A, B)$  has index  $\nu$  for some  $\nu \geq 0$  if and only if the matrix pencil  $\lambda A + B$  is regular, and that  $\nu$  is exactly the index of the matrix pencil  $\lambda A + B$ .  $\square$

From now on, when referring to the pair  $(A, B)$  with index  $\nu \geq 0$ , it will always be implicitly assumed that the reduction was possible up to and including step  $\nu$  (and hence beyond); that is, for the time being, that (2.1) and (2.2) hold for  $0 \leq j \leq \nu$  (and hence for  $j \geq \nu + 1$ ).

Suppose now that the pair  $(A, B)$  has index  $\nu$ , and consider the DAE (1.1) with  $b \in (\mathcal{D}'(\mathbf{R}))^n$ . The condition (2.2) is equivalent to the invertibility of  $[A_j(t) \oplus B_j(t)]^T$  and hence to  $\ker A_j^T(t) \cap \ker B_j^T(t) = \{0\}$ , or, equivalently, to the invertibility of  $A_j(t)A_j(t)^T + B_j(t)B_j(t)^T$ . We now define sequences  $u_0, \dots, u_{\nu-1}$  and  $b_0, \dots, b_\nu$  of distributions as follows: Set  $b_0 = b$  and, generally, if  $b_j$ ,  $0 \leq j \leq \nu$  is known, construct  $u_j$  by multiplying the distribution  $b_j$  by the  $C^\infty$  operator  $B_j^T(A_j A_j^T + B_j B_j^T)^{-1}$ ; that is,

$$(2.4) \quad u_j = B_j^T (A_j A_j^T + B_j B_j^T)^{-1} b_j,$$

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Moreover, for  $0 \leq j \leq \nu - 1$ , define

$$(2.5) \quad b_{j+1} = D_j(b_j - Bu_j - A\dot{u}_j),$$

and

$$(2.6) \quad \Gamma_{\nu-1} = C_0 C_1 \cdots C_{\nu-1} \in C^\infty(\mathbf{R}; \mathcal{L}(\mathbf{R}^{r_{\nu-1}}, \mathbf{R}^n))$$

$$(2.7) \quad v_{\nu-1} = u_0 + C_0 u_1 + C_0 C_1 u_2 + \cdots + C_0 \cdots C_{\nu-2} u_{\nu-1} \in (\mathcal{D}'(\mathbf{R}))^n.$$

In [RR93a] it is shown that a distribution  $x \in (\mathcal{D}'(\mathbf{R}))^n$  solves (1.1) if and only if  $x$  has the form

$$(2.8) \quad x = \Gamma_{\nu-1} x_\nu + v_{\nu-1},$$

with  $\Gamma_{\nu-1}$  and  $v_{\nu-1}$  given by (2.6) and (2.7), respectively, and  $x_\nu \in (\mathcal{D}'(\mathbf{R}))^{r_{\nu-1}}$  is a solution of the ODE

$$(2.9) \quad \dot{x}_\nu + A_\nu^{-1} B_\nu x_\nu = A_\nu^{-1} b_\nu.$$

Naturally, the equivalence between (1.1) and (2.9) via (2.8) is true, in particular, for classical solutions; that is, when (say)  $b \in C^\infty(\mathbf{R}; \mathbf{R}^n)$ . Then also  $u_j$  and  $b_j$ , defined by (2.4) and (2.5), respectively, are of class  $C^\infty$ , and so is  $v_{\nu-1}$  in (2.7). In this case, (2.8) also transforms initial value problems for (1.1) into initial value problems for (2.9). In fact,  $x$  solves (1.1) under the initial condition  $x(t_0) = x_0$  for fixed  $t_0 \in \mathbf{R}$  if and only if  $x_0$  verifies

$$(2.10) \quad x_0 = \Gamma_{\nu-1}(t_0) x_{\nu 0} + v_{\nu-1}(t_0),$$

for some  $x_{\nu 0} \in \mathbf{R}^{r_{\nu-1}}$  which, of course, is necessarily unique by the injectivity of  $\Gamma_{\nu-1}(t_0)$ . Such values  $x_0$  are called *consistent* with the DAE (1.1) at  $t_0$ . Evidently, if  $x$  solves (1.1) then the values  $x(t) \in \mathbf{R}^n$  are consistent with (1.1) at  $t$ ,  $\forall t \in \mathbf{R}$ . Moreover, initial value

problems for (1.1) with consistent initial values at the given point  $t_0$  have a unique ( $C^\infty$ ) solution while initial value problems with non-consistent initial values have no (classical) solution.

It is an interesting fact that the condition (2.1) is essentially superfluous if  $A$  and  $B$  are *analytic* (see [RR93a]), partly because in that case (2.1) automatically holds with  $r_j = \max_{t \in \mathbb{R}} \text{rank } A_j(t)$ , except perhaps at points of a subset  $\mathcal{S}_j$  consisting only of isolated points in  $\mathbb{R}$ . For  $t \in \mathcal{S}_j$ , we have  $\dim \text{rge } A_j(t) < r_j$ , but it turns out that an “extended range” of  $A_j(t)$ , denoted by  $\text{ext rge } A_j(t)$ , can be defined with the properties that  $\text{ext rge } A_j(t) \supset \text{rge } A_j(t)$  and  $\dim \text{ext rge } A_j(t) = r_j$ ,  $\forall t \in \mathbb{R}$  (and hence  $\text{ext rge } A_j(t) = \text{rge } A_j(t)$ ,  $\forall t \in \mathbb{R} \setminus \mathcal{S}_j$ ). This allows for the construction of parametrized families  $P_j$ ,  $C_j$  and  $D_j$  as before, except that “ $\text{ext rge } A_j(t)$ ” now replaces “ $\text{rge } A_j(t)$ ” everywhere. Thus, assuming only that (2.2) holds for all indices  $j$ , we can still construct a reduction  $(A_{j+1}, B_{j+1})$  of  $(A_j, B_j)$ , and the index  $\nu$  of  $(A, B)$  is defined as before. But now, we have only  $A_\nu(t) \in GL(\mathbb{R}^{r_\nu-1})$  for  $t \in \mathbb{R} \setminus \mathcal{S}_\nu$ , and (1.1) reduces to (2.9) via (2.8) only if  $A_\nu(t) \in GL(\mathbb{R}^{r_\nu-1})$  for every  $t \in \mathbb{R}$ . Thus, in this case, the invertibility of  $A_\nu(t)$  for all  $t$  is no longer guaranteed and must be assumed independently. As a result, future reference to pairs  $(A, B)$  of index  $\nu$  “with invertible  $A_\nu(t)$  for every  $t \in \mathbb{R}$ ” should not be viewed as a redundancy but, rather, as a reminder that condition (2.1) can be dropped if  $A$  and  $B$  are analytic but that then invertibility of  $A_\nu(t)$  is no longer guaranteed to hold for all  $t$ .

All indicated results extend verbatim to the case when  $\mathbb{R}$  is replaced by an arbitrary interval  $\mathcal{J}$  where distributions in  $\mathcal{J}$  are now understood to be distributions in  $\dot{\mathcal{J}}$ . If  $\mathcal{J}$  contains one of its endpoints, initial value problems for (1.1) with a consistent initial value at that endpoint can be considered in the classical setting.

### 3. Initial value problems with discontinuous right sides.

In [HS83], Hautus and Silverman introduced the class  $\mathcal{C}_{\text{imp}}$  of “impulsive-smooth” distributions in  $[0, \infty)$ . We first need a straightforward variant of this concept for distributions in  $\mathbb{R}$ . Throughout the remainder of this presentation,  $\mathbb{R}^*$  denotes  $\mathbb{R} \setminus \{0\}$ .



**Definition 3.1.** The distribution  $x \in \mathcal{D}'(\mathbb{R})$  is said to be impulsive-smooth,  $x \in \mathcal{C}_{\text{imp}}(\mathbb{R}^*)$  for short, if there are functions  $\varphi, \psi \in C^\infty(\mathbb{R})$  such that  $x - \varphi H - \psi(1 - H)$  is a distribution with support  $\{0\}$  where  $H$  denotes the Heaviside function.

If  $x \in \mathcal{C}_{\text{imp}}(\mathbb{R}^*)$  and  $\varphi_1, \varphi_2, \psi_1, \psi_2 \in C^\infty(\mathbb{R})$  are such that  $x - \varphi_i H - \psi_i(1 - H)$  is a distribution with support  $\{0\}$ ,  $i = 1, 2$ , then  $(\varphi_1 - \varphi_2)H + (\psi_1 - \psi_2)(1 - H)$  is a distribution with support  $\{0\}$  and hence must be a linear combination of the Dirac  $\delta$  and its derivatives. But, since it is also a function, it must be 0; that is,  $\varphi_1 H + \psi_1(1 - H) = \varphi_2 H + \psi_2(1 - H)$ . This shows that  $x - \varphi_i H - \psi_i(1 - H)$  is identical for  $i = 1$  and  $i = 2$ , and hence can be called the impulsive part of  $x$ , denoted by  $x_{\text{imp}}$ .

Therefore, for given  $x \in \mathcal{C}_{\text{imp}}(\mathbb{R}^*)$  the difference  $x - x_{\text{imp}}$  has the form  $\varphi H + \psi(1 - H)$  with  $\varphi, \psi \in C^\infty(\mathbb{R})$ . Of course,  $\varphi$  and  $\psi$  are not uniquely determined by this condition, but  $\psi|_{(-\infty, 0]}$  and  $\varphi|_{[0, \infty)}$  are. Thus, there is no ambiguity in setting

$$x_- = \psi|_{(-\infty, 0]}, \quad x_+ = \varphi|_{[0, \infty)}.$$

With this definition,  $x_- \in C^\infty((-\infty, 0])$  and  $x_+ \in C^\infty([0, \infty))$ , and extending  $x_-$  by 0 for  $t > 0$  and  $x_+$  by 0 for  $t < 0$ , we may write

$$(3.1) \quad x = x_- + x_+ + x_{\text{imp}},$$

where each of the three terms on the right side is uniquely determined by  $x$ . Conversely, given  $x_- \in C^\infty((-\infty, 0])$  and  $x_+ \in C^\infty([0, \infty))$  and a distribution  $x_{\text{imp}}$  with support  $\{0\}$ , equation (3.1) defines an element  $x$  of  $\mathcal{C}_{\text{imp}}(\mathbb{R}^*)$ .

**Remark 3.1.** Despite the terminology "impulsive-smooth", it should be kept in mind that for  $x \in \mathcal{C}_{\text{imp}}(\mathbb{R}^*)$ ,  $x - x_{\text{imp}}$  is not a smooth function in  $\mathbb{R}$  since it may have a jump at 0. But its restrictions  $x_-$  and  $x_+$  to  $(-\infty, 0)$  and  $(0, \infty)$ , respectively, extend as smooth functions in  $(-\infty, 0]$  and  $[0, \infty)$ , respectively.  $\square$

Three trivial but essential properties of impulsive-smooth distributions are the following:

(i) Every  $x \in \mathcal{C}_{\text{imp}}(\mathbb{R}^*)$  may be assigned a value at every point  $t \neq 0$ , namely  $x(t) = x_-(t)$

if  $t < 0$  and  $x(t) = x_+(t)$  if  $t > 0$ . (ii) The derivative and the primitives (in the sense of distributions) of  $x \in C_{\text{imp}}(\mathbf{R}^*)$  are themselves in  $C_{\text{imp}}(\mathbf{R}^*)$ . (iii)  $C_{\text{imp}}(\mathbf{R}^*)$  is both a vector space over  $\mathbf{R}$  and a  $C^\infty(\mathbf{R})$ -module. In fact, if  $x \in C_{\text{imp}}(\mathbf{R}^*)$  and (3.1) is used, then we have

$$(3.2) \quad \alpha x = \alpha x_- + \alpha x_+ + \sum_{i=0}^k \left( \sum_{j=0}^{k-i} (-1)^j \binom{j+i}{j} \alpha^{(j)}(0) \lambda_{i+j} \right) \delta^{(i)}, \quad \forall \alpha \in C^\infty(\mathbf{R}),$$

whenever  $x_{\text{imp}} = \sum_{i=0}^k \lambda_i \delta^{(i)}$ ,  $\lambda_i \in \mathbf{R}$ ,  $0 \leq i \leq k$ . The above properties, including (3.1), have an immediate generalization to elements of  $C_{\text{imp}}^n(\mathbf{R}^*) \equiv (C_{\text{imp}}(\mathbf{R}^*))^n$ . In particular, for  $x \in C_{\text{imp}}^n(\mathbf{R}^*)$ , and  $M \in C^\infty(\mathbf{R}; \mathcal{L}(\mathbf{R}^n, \mathbf{R}^m))$ , we have  $Mx \in C_{\text{imp}}^m(\mathbf{R}^*)$  and

$$(3.3) \quad Mx = Mx_- + Mx_+ + \sum_{i=0}^k \left( \sum_{j=0}^{k-i} (-1)^j \binom{j+i}{j} M^{(j)}(0) \lambda_{i+j} \right) \delta^{(i)},$$

whenever

$$(3.4) \quad x_{\text{imp}} = \sum_{i=0}^k \lambda_i \delta^{(i)}, \quad \lambda_i \in \mathbf{R}^n, \quad 0 \leq i \leq k.$$

**Definition 3.2.** Let  $x \in C_{\text{imp}}^n(\mathbf{R}^*)$ , so that there exists a unique decomposition (3.4). The impulse order of  $x$ , denoted by  $\text{iord}(x)$ , is defined as follows:

(i) If  $\lambda_i = 0$ ,  $0 \leq i \leq k$  and  $x_+(0) = x_-(0)$ , and hence  $x \in C^0(\mathbf{R}; \mathbf{R}^n)$ , then set

$$\text{iord}(x) = -m - 2,$$

where  $0 \leq m \leq \infty$  is the largest integer such that  $x \in C^m(\mathbf{R}; \mathbf{R}^n)$ .

(ii) If  $\lambda_i = 0$ ,  $0 \leq i \leq k$  and  $x_+(0) \neq x_-(0)$  (and hence  $x$  has a discontinuity at the origin), then set

$$\text{iord}(x) = -1.$$

(iii) If  $\lambda_i \neq 0$  for some  $0 \leq i \leq k$  then set

$$\text{iord}(x) = \max\{i : 0 \leq i \leq k, \lambda_i \neq 0\}.$$

**Remark 3.2:** Let  $M \in C^\infty(\mathbf{R}; \mathcal{L}(\mathbf{R}^n, \mathbf{R}^m))$ . By (3.3) we have  $\text{iord}(Mx) \leq \text{iord}(x)$  and equality holds if  $n = m$  and  $M(0)$  is invertible.  $\square$

The following lemma provides some more precise preliminary results about the primitives in the sense of distributions of the elements of  $C_{\text{imp}}^n(\mathbf{R}^*)$ .

**Lemma 3.1.** (i) Let  $f \in C_{\text{imp}}^n(\mathbf{R}^*)$  have impulse order  $k \in \mathbf{Z} \cup \{-\infty\}$  and let  $y \in (\mathcal{D}'(\mathbf{R}))^n$  be such that  $\dot{y} = f$ . Then,  $y \in C_{\text{imp}}^n(\mathbf{R}^*)$  and  $y$  has impulse order  $k - 1$ .

(ii) Let  $f \in C_{\text{imp}}^n(\mathbf{R}^*)$  and let  $x_0 \in \mathbf{R}^n$  and  $t_0 \in \mathbf{R}^*$  be given. Then, there is a unique  $y \in C_{\text{imp}}^n(\mathbf{R}^*)$  such that  $\dot{y} = f$  and exactly one of the following conditions holds

$$(3.5) \quad (a) \ y(t_0) = x_0, \quad (b_+) \ y_+(0) = x_0, \quad (b_-) \ y_-(0) = x_0.$$

(iii) Let the sequence  $f^\ell \in C_{\text{imp}}^n(\mathbf{R}^*)$ ,  $\ell \geq 1$ , and  $x_0 \in \mathbf{R}^n$  and  $t_0 \in \mathbf{R}^*$  be given. Suppose that there are an open interval  $I_{t_0}$  about  $t_0$  and some  $f \in C_{\text{imp}}^n(\mathbf{R}^*)$  such that

$$(3.6) \quad f^\ell|_{I_{t_0}} = f|_{I_{t_0}}, \quad \forall \ell \geq 1,$$

as distributions in  $I_{t_0}$  (i.e. as functions if  $0 \notin I_{t_0}$ ) and that

$$(3.7) \quad \lim_{\ell \rightarrow \infty} f^\ell = f \quad \text{in } (\mathcal{D}'(\mathbf{R}))^n.$$

Let  $y^\ell \in C_{\text{imp}}^n(\mathbf{R}^*)$  and  $y \in C_{\text{imp}}^n(\mathbf{R}^*)$  be such that  $\dot{y}^\ell = f^\ell$ ,  $y^\ell(t_0) = x_0$  and  $\dot{y} = f$ ,  $y(t_0) = x_0$  (see (ii) above). Then, we have

$$(3.8) \quad \lim_{\ell \rightarrow \infty} y^\ell = y \quad \text{in } (\mathcal{D}'(\mathbf{R}))^n.$$

(A similar result holds if  $t_0 = 0$  and  $y^\ell, y$  are characterized by  $\dot{y}^\ell = f^\ell$ ,  $y_\pm^\ell(0) = x_0$  and  $\dot{y} = f$ ,  $y_\pm(0) = x_0$ .)

*Proof:* (i) Any two primitives (in the sense of distributions) of an element of  $(\mathcal{D}'(\mathbf{R}))^n$  differ from a constant vector of  $\mathbf{R}^n$ , and addition of a constant vector does not affect membership

to  $C_{\text{imp}}^n(\mathbf{R}^*)$  nor the impulse order. Thus, it suffices to show that  $f$  has one primitive in  $C_{\text{imp}}^n(\mathbf{R}^*)$  with impulse order  $k-1$ .

Write  $f = f_+ + f_- + f_{\text{imp}}$  with  $f_{\text{imp}} = \sum_{i=0}^k \mu_i \delta^{(i)}$ ,  $\mu_i \in \mathbf{R}^n$ ,  $\mu_k \neq 0$  and choose  $t_0 \in \mathbf{R}$ . Since the function  $f_+ + f_-$  is locally integrable, set  $\tilde{y} = \int_{t_0}^t (f_+ + f_-)(s) ds$ , so that  $\tilde{y} \in C^\infty((-\infty, 0]; \mathbf{R}^n) \cap C^\infty([0, \infty); \mathbf{R}^n) \cap C^0(\mathbf{R}; \mathbf{R}^n) \cap C_{\text{imp}}^n(\mathbf{R}^*)$  and  $\tilde{y}$  is a primitive of  $f_+ + f_-$  in the sense of distributions. Furthermore, it is obvious that  $\tilde{y} \in C^{m+1}(\mathbf{R}; \mathbf{R}^n)$  whenever  $f_+ + f_- \in C^m(\mathbf{R}; \mathbf{R}^n)$ ,  $0 \leq m \leq \infty$ . Thus,  $\text{iord}(\tilde{y}) = \text{iord}(f_+ + f_-) - 1 \leq -2$ . In particular,  $\tilde{y}$  is a primitive of  $f$  with impulse order  $k-1$  when  $k < 0$  since  $f = f_+ + f_-$  in this case.

Suppose now that  $k \geq 0$  and set  $y_{\text{imp}} = \sum_{i=1}^k \mu_i \delta^{i-1}$ , so that  $\mu_0 H + y_{\text{imp}}$  is a primitive of  $f_{\text{imp}}$ . Evidently,  $\mu_0 H + y_{\text{imp}} \in C_{\text{imp}}^n(\mathbf{R}^*)$  and  $\text{iord}(\mu_0 H + y_{\text{imp}}) = k-1 \geq -1$  since  $k \geq 0$  and  $\mu_k \neq 0$ . Thus,  $y = \tilde{y} + \mu_0 H + y_{\text{imp}} \in C_{\text{imp}}^n(\mathbf{R}^*)$  is a primitive of  $f$  and  $\text{iord}(y) = k-1$  since  $\text{iord}(\tilde{y}) \leq -2 < -1 \leq \text{iord}(\mu_0 H + y_{\text{imp}})$ .

(ii) The primitive of  $f$  obtained in (i) verifies  $y_- = \tilde{y}_-$  and  $y_+ = \tilde{y}_+ + \mu_0$ . Since  $\tilde{y}$  is continuous and  $\tilde{y}(t_0) = 0$ , this yields  $y_-(t_0) = 0$  and  $y_+(t_0) = \mu_0$ . Every other primitive of  $f$  (still denoted by  $y$  for simplicity of notation) is uniquely characterized by a vector  $\lambda_0 \in \mathbf{R}^n$  and verifies  $y_-(t_0) = \lambda_0$ ,  $y_+(t_0) = \mu_0 + \lambda_0$ . As a result,  $\lambda_0 = x_0$  (resp.  $\lambda_0 = x_0 - \mu_0$ ) is the only possible choice yielding  $y_-(t_0) = x_0$  (resp.  $y_+(t_0) = x_0$ ). Letting  $t_0 = 0$ , we obtain existence and uniqueness of  $y \in C_{\text{imp}}^n(\mathbf{R}^*)$  such that  $\dot{y} = f$  and either (3.5)(b<sub>+</sub>) or (3.5)(b<sub>-</sub>) holds. Next, letting  $t_0 \neq 0$  and observing that  $y(t_0) = y_-(t_0)$  if  $t_0 < 0$  and  $y(t_0) = y_+(t_0)$  if  $t_0 > 0$ , we obtain existence and uniqueness of  $y \in C_{\text{imp}}^n(\mathbf{R}^*)$  such that  $\dot{y} = f$  and  $y(t_0) = x_0$ .

(iii) To begin with, let us briefly recall how primitives of distributions are defined: Let  $\theta \in \mathcal{D}(\mathbf{R})$  be such that  $\int_{\mathbf{R}} \theta = 1$ . For  $\varphi \in (\mathcal{D}(\mathbf{R}))^n$ , there is a unique  $\psi \in (\mathcal{D}(\mathbf{R}))^n$  such that  $\dot{\psi} = \varphi - \theta \int_{\mathbf{R}} \varphi$  and the correspondence  $\varphi \mapsto \psi$  is continuous for the usual topology of  $(\mathcal{D}(\mathbf{R}))^n$ . Note also that  $\text{supp } \psi \subset \text{supp } \varphi \cup \text{supp } \theta$ . Given  $T = (T_1, \dots, T_n) \in (\mathcal{D}'(\mathbf{R}))^n$ , the formula

$$(3.9) \quad \langle S, \varphi \rangle = -\langle T, \psi \rangle + c \cdot \int_{\mathbf{R}} \varphi,$$

with  $c \in \mathbb{R}^n$  and the dot denoting the usual inner product of  $\mathbb{R}^n$ , defines  $S$  as a distribution with values in  $\mathbb{R}^n$  and shows that  $\dot{S} = T$ , and all the primitives of  $T$  are of the form (3.9) for some  $c \in \mathbb{R}^n$ .

In general, the formula (3.9) does not permit us to assign a value  $S(t)$  to  $S$  for any  $t \in \mathbb{R}$ . But suppose that there are  $t_0 \in \mathbb{R}$  and an open interval  $I_{t_0}$  about  $t_0$  such that  $T|_{I_{t_0}}$  is (say) a  $C^\infty$  function, whence  $\langle T, \varphi \rangle = \int_{I_{t_0}} T \cdot \varphi = \int_{\mathbb{R}} T \cdot \varphi$ , for all  $\varphi \in (\mathcal{D}(I_{t_0}))^n$ . We may choose  $\theta$  such that  $\text{supp } \theta \subset I_{t_0}$  and then, for  $t \in I_{t_0}$ , we may define  $S_0(t) = \int_{t_0}^t T(s) ds$ ; that is,  $S = (S_{01}, \dots, S_{0n})$  with  $S_{0i}(t) = \int_{t_0}^t T_i(s) ds$ . In (3.9) let  $c = (c_1, \dots, c_n)$  with

$$(3.10) \quad c_i = \int_{\mathbb{R}} S_{0i}(t) \theta(t) dt, \quad 1 \leq i \leq n.$$

This makes sense since  $\text{supp } \theta \subset I_{t_0}$  and  $S_{0i}(t)$  is defined for  $t \in I_{t_0}$ . Let  $\varphi \in (\mathcal{D}(I_{t_0}))^n$ , whence  $\psi \in (\mathcal{D}(I_{t_0}))^n$  since  $\text{supp } \theta, \text{supp } \varphi \subset I_{t_0}$ . As  $T = \dot{S}_0$  in  $(\mathcal{D}'(I_{t_0}))^n$ , relation (3.9) reads

$$\begin{aligned} \langle S, \varphi \rangle &= -\langle \dot{S}_0, \psi \rangle + c \cdot \int_{\mathbb{R}} \varphi \\ &= \langle S_0, \dot{\psi} \rangle + c \cdot \int_{\mathbb{R}} \varphi = \langle S_0, \varphi - \theta \int_{\mathbb{R}} \varphi \rangle + c \cdot \int_{\mathbb{R}} \varphi \\ &= \langle S_0, \varphi \rangle - \langle S_0, \theta \int_{\mathbb{R}} \varphi \rangle + c \cdot \int_{\mathbb{R}} \varphi. \end{aligned}$$

But

$$\begin{aligned} \langle S_0, \theta \int_{\mathbb{R}} \varphi \rangle &= \int_{I_{t_0}} S_0 \cdot \theta \int_{\mathbb{R}} \varphi \\ &= \int_{\mathbb{R}} S_0 \cdot \theta \int_{\mathbb{R}} \varphi = \sum_{i=1}^n \left( \int_{\mathbb{R}} S_{0i} \theta \right) \int_{\mathbb{R}} \varphi_i = c \cdot \int_{\mathbb{R}} \varphi, \end{aligned}$$

by definition of  $c$  in (3.10). Thus,  $\langle S, \varphi \rangle = \langle S_0, \varphi \rangle$ , for all  $\varphi \in (\mathcal{D}(I_{t_0}))^n$ , i.e.,  $S|_{I_{t_0}} = S_0$ . Because  $S_0$  is a function and vanishes at  $t_0$ , it follows that  $S$  in (3.9) may be referred to as the primitive of  $T$  vanishing at  $t_0$  when  $c$  is chosen as in (3.10) (and  $\theta$  verifies  $\text{supp } \theta \subset I_{t_0}$ ). The independence of this definition from the choice of  $\theta$  is easily seen: If  $S, \hat{S}$  corresponds

to two such choices, we have  $\tilde{S} = S + c, c \in \mathbb{R}^n$  since both  $S$  and  $\tilde{S}$  are primitives of  $T$ , and  $c = 0$  from  $\tilde{S}|_{t_0} = S_0 = S|_{t_0}$ . As a result, given  $x_0 \in \mathbb{R}^n$ ,  $S + x_0$  may be referred to as the primitive of  $T$  verifying  $S(t_0) = x_0$ .

Now, with  $T$  as above, let  $T^\ell \in (\mathcal{D}'(\mathbb{R}))^n$ ,  $\ell \geq 1$ , be a sequence such that  $T|_{t_0} = T^\ell|_{t_0}$ . This assumption ensures that  $T^\ell$ ,  $\ell \geq 1$ , as well as  $T$  define the same vector  $c$  in (3.10). With this choice of  $c$ , the distribution  $S^\ell \in (\mathcal{D}'(\mathbb{R}))^n$  obtained by replacing  $T$  by  $T^\ell$  in (3.9) is the primitive of  $T^\ell$  vanishing at  $t_0$ , and, under the assumption  $\lim_{\ell \rightarrow \infty} T^\ell = T$  in  $(\mathcal{D}'(\mathbb{R}))^n$ , it is then obvious that for  $\varphi \in (\mathcal{D}(\mathbb{R}))^n$  we have  $\lim_{\ell \rightarrow \infty} \langle S^\ell, \varphi \rangle = \langle S, \varphi \rangle$ , i.e.  $\lim_{\ell \rightarrow \infty} S^\ell = S$  in  $(\mathcal{D}'(\mathbb{R}))^n$ . In turn, this implies that  $\lim_{\ell \rightarrow \infty} S^\ell + x_0 = S + x_0$  for  $x_0 \in \mathbb{R}^n$ .

It should be clear that part (iii) of the lemma follows from the above considerations with  $T = f$  and  $T^\ell = f^\ell$ , so that  $S + x_0 = y$  and  $S^\ell + x_0 = y^\ell$ . The proof that a result similar to (3.8) holds when  $t_0 = 0$  and  $y^\ell, y$  are characterized by  $\dot{y}^\ell = f^\ell$ ,  $y_\pm^\ell(0) = x_0$  and  $\dot{y} = f$ ,  $y_\pm(0) = x_0$ , easily follows from the above considerations with  $T = f_+ + f_-$ ,  $T^\ell = f_+^\ell + f_-^\ell$ , and the remark that  $f_{\text{imp}}^\ell = f_{\text{imp}}$ , for all  $\ell \geq 1$ , since  $f^\ell$  and  $f$  coincide as distributions in some open interval about the origin by hypothesis. Details are left to the reader.  $\square$

**Remark 3.3:** Because of condition (3.6), Lemma 3.1 (iii) gives an unusual result about continuous dependence for initial value problems. The incorrectness of this result under condition (3.7) alone can be seen even in the case when  $n = 1$  and  $f, f^\ell \in C^\infty(\mathbb{R})$ . In fact, by the theory of Fourier series, the sequences  $\ell^\alpha \cos \ell t$  and  $\ell^\alpha \sin \ell t$  tend to 0 in  $\mathcal{D}'(\mathbb{R})$  for every  $\alpha \in \mathbb{R}$ . In particular, if  $f = 0$ ,  $f^\ell(t) = \ell \sin \ell t$ , we have  $\lim_{\ell \rightarrow \infty} f^\ell = f$  in  $\mathcal{D}'(\mathbb{R})$  as in (3.7). Choosing  $t_0 = 0$ , we find  $y = 0$ ,  $y^\ell(t) = 1 - \cos \ell t$  in Lemma 3.1 (iii). But then,  $\lim_{\ell \rightarrow \infty} y^\ell = 1 \neq y$  in  $\mathcal{D}'(\mathbb{R})$  and (3.8) fails to hold.  $\square$

Lemma 3.1 has a direct application to initial value problems for the ODE

$$(3.11) \quad \dot{x} + Mx = f,$$

considered in the following theorem:

**Theorem 3.1.** Let  $M \in C^\infty(\mathbb{R}; \mathcal{L}(\mathbb{R}^n))$  and let  $f \in C_{\text{imp}}^n(\mathbb{R}^*)$  have impulse order  $k \in \mathbb{Z} \cup \{-\infty\}$ . Then

(i) The solutions  $x \in (\mathcal{D}'(\mathbb{R}))^n$  of the ODE (3.11) belong to  $C_{\text{imp}}^n(\mathbb{R}^*)$  and have impulse order  $k - 1$ .

(ii) For given  $x_0 \in \mathbb{R}^n$  and  $t_0 \in \mathbb{R}^*$ , the ODE (3.11) together with one of the initial conditions

$$(3.12) \quad (a) \quad x(t_0) = x_0, \quad (b_+) \quad x_+(0) = x_0, \quad (b_-) \quad x_-(0) = x_0,$$

has a unique solution  $x \in C_{\text{imp}}^n(\mathbb{R}^*)$ , but in the cases (3.12)( $b_+$ ) and (3.12)( $b_-$ ) the solutions corresponding to  $x_+(0) = x_0$  and  $x_-(0) = x_0$  need not be the same.

(iii) Let the sequence  $f^\ell \in C_{\text{imp}}^n(\mathbb{R}^*)$ ,  $\ell \geq 1$ , and  $x_0 \in \mathbb{R}^n$  and  $t_0 \in \mathbb{R}^*$  be given. Suppose that there is an open interval  $I_{t_0}$  about  $t_0$  such that

$$(3.13) \quad f^\ell|_{I_{t_0}} = f|_{I_{t_0}}, \quad \forall \ell \geq 1,$$

as distributions in  $I_{t_0}$ , (i.e., as functions if  $0 \notin I_{t_0}$ ) and

$$(3.14) \quad \lim_{\ell \rightarrow \infty} f^\ell = f \quad \text{in } (\mathcal{D}'(\mathbb{R}))^n.$$

Let  $x$  (resp.  $x^\ell$ )  $\in C_{\text{imp}}^n(\mathbb{R}^*)$  denote the unique solution of the ODE (3.11) (resp. (3.11) with  $f$  replaced by  $f^\ell$ ) verifying  $x(t_0) = x_0$  (resp.  $x^\ell(t_0) = x_0$ ) whose existence is ensured by part (ii) of the theorem. Then

$$(3.15) \quad \lim_{\ell \rightarrow \infty} x^\ell = x \quad \text{in } (\mathcal{D}'(\mathbb{R}))^n.$$

(A similar result holds if  $t_0 = 0$  and the initial condition for  $x^\ell$ ,  $x$  is chosen as  $x_\pm^\ell(0) = x_0$  and  $x_\pm(0) = x_0$ , respectively.)

*Proof:* (i) Fix  $t_0 \in \mathbb{R}$  and denote by  $U \in C^\infty(\mathbb{R}; \mathcal{L}(\mathbb{R}^n))$  the solution of the initial value problem  $\dot{U} + MU = 0$ ,  $U(t_0) = I$ . It is well known that  $U(t) \in GL(\mathbb{R}^n)$  for all  $t \in \mathbb{R}$ .

Then,  $x \in (\mathcal{D}'(\mathbb{R}))^n$  solves (3.11) if and only if  $y = U^{-1}x \in (\mathcal{D}'(\mathbb{R}))^n$  solves the equation  $\dot{y} = U^{-1}f$ . Since  $f \in C_{\text{imp}}^n(\mathbb{R}^*)$ , we have  $U^{-1}f \in C_{\text{imp}}^n(\mathbb{R}^*)$  and  $\text{iord}(U^{-1}f) = \text{iord}(f) = k$  from Remark 3.2. Next, by Lemma 3.1 (i) the solutions of  $\dot{y} = U^{-1}f$  are in  $C_{\text{imp}}^n(\mathbb{R}^*)$  and  $\text{iord}(x) = \text{iord}(y) = k - 1$  by another application of Remark 3.2.

(ii) Since  $U(t_0) = I$ , we have  $x(t_0) = x_0$  (resp.  $t_0 = 0$  and  $x_{\pm}(0) = x_0$ ) if and only if  $y(t_0) = x_0$  (resp.  $t_0 = 0$  and  $y_{\pm}(0) = x_0$ ). Existence and uniqueness of  $x$  thus follows from existence and uniqueness of  $y$  ensured by Lemma 3.1 (ii).

(iii) From conditions (3.13) and (3.14) and using the continuity of multiplication of distributions by  $C^\infty$  matrix-valued functions, we infer that  $U^{-1}f|_{t_0}^\ell = U^{-1}f|_{t_0}$  and  $\lim_{\ell \rightarrow \infty} U^{-1}f^\ell = U^{-1}f$  in  $(\mathcal{D}'(\mathbb{R}))^n$ . Denoting by  $y^\ell$  the solution of  $\dot{y}^\ell = U^{-1}f^\ell$ ,  $y^\ell(0) = x_0$ , we find that  $\lim_{\ell \rightarrow \infty} y^\ell = y$  in  $(\mathcal{D}'(\mathbb{R}))^n$  by Lemma 3.1 (iii). Thus,  $\lim_{\ell \rightarrow \infty} x^\ell = x$  in  $(\mathcal{D}'(\mathbb{R}))^n$  since  $x^\ell = Uy^\ell$ ,  $x = Uy$  and multiplication by  $U$  is continuous.  $\square$

**Remark 3.4:** Let  $f_{\text{imp}} = \sum_{i=0}^k \mu_i \delta^{(i)}$  with  $\mu_k \neq 0$ . From Theorem 3.1, we have  $x_{\text{imp}} = \sum_{i=0}^{k-1} \lambda_i \delta^{(i)}$  for every solution  $x \in C_{\text{imp}}^n(\mathbb{R}^*)$  of  $\dot{x} + Mx = f$ . Comparing impulsive parts and using (3.3) - (3.4) yields

$$\begin{cases} \lambda_{k-1} = \mu_k, \\ \lambda_{i-1} + \sum_{j=0}^{k-i-1} (-1)^j \binom{j+i}{j} M^{(j)}(0) \lambda_{i+j} = \mu_i, \quad 1 \leq i \leq k-1, \\ x_+(0) - x_-(0) + \sum_{i=0}^{k-1} (-1)^i M^{(i)}(0) \lambda_i = \mu_0. \end{cases}$$

By inverting these formulas, we find  $\lambda_0, \dots, \lambda_{k-1}$  (depending only upon  $\mu_1, \dots, \mu_k$ ) as well as  $x_+(0) - x_-(0)$ . Thus, both  $x_{\text{imp}}$  and  $x_+(0) - x_-(0)$  are calculable and depend solely upon  $f_{\text{imp}}$ .  $\square$

We now focus on initial value problems for the DAE

$$(3.16) \quad A\dot{x} + Bx = b,$$

where  $A, B \in C^\infty(\mathbb{R}; \mathcal{L}(\mathbb{R}^n))$  and  $b \in C_{\text{imp}}^n(\mathbb{R}^*)$ . Under the assumption that the pair  $(A, B)$



has index  $\nu \geq 0$  in  $\mathbf{R}$  and that  $A_\nu(t)$  is invertible for every  $t \in \mathbf{R}$  (see Section 2), the DAE's

$$(3.17_-) \quad A(t)\dot{x}_- + B(t)x_- = b_-(t) \quad \text{in } (-\infty, 0),$$

and

$$(3.17_+) \quad A(t)\dot{x}_+ + B(t)x_+ = b_+(t) \quad \text{in } (0, \infty),$$

have coefficients and right-hand sides of class  $C^\infty$  in  $(-\infty, 0]$  and  $[0, \infty)$ , respectively. As a consequence, it makes sense to speak of values  $x_0 \in \mathbf{R}^n$  which are consistent with (3.17\_-) (resp. (3.17\_+)) at a point  $t_0 \leq 0$  (resp.  $t_0 \geq 0$ ) in the sense of Section 2.

**Theorem 3.2.** *Let the pair  $(A, B)$ ,  $A, B \in C^\infty(\mathbf{R}; \mathcal{L}(\mathbf{R}^n))$  have index  $\nu \geq 0$ , and with the notation of Section 2, suppose that  $A_\nu(t)$  is invertible for every  $t \in \mathbf{R}$ . If  $b \in C_{\text{imp}}^n(\mathbf{R}^*)$  has impulse order  $k \in \mathbf{Z} \cup \{-\infty\}$ , then, the solutions  $x \in (\mathcal{D}'(\mathbf{R}))^n$  of the DAE (3.16) are in  $C_{\text{imp}}^n(\mathbf{R}^*)$  and have impulse order at most  $k + \nu - 1$ . Moreover,*

(i) *if  $t_0 < 0$  (resp.  $t_0 > 0$ ) and  $x_0 \in \mathbf{R}^n$  is consistent with the DAE (3.17\_-) (resp. (3.17\_+)) at  $t_0$ , the initial value problem*

$$(3.18) \quad A\dot{x} + Bx = b, \quad x(t_0) = x_0,$$

*has a unique solution  $x \in C_{\text{imp}}^n(\mathbf{R}^*)$ . Furthermore, if  $I_{t_0}$  is an open interval about  $t_0$  and  $b^\ell \in C_{\text{imp}}^n(\mathbf{R}^*)$  is a sequence such that  $b^\ell|_{I_{t_0}} = b|_{I_{t_0}}$ , for all  $\ell \geq 1$ , as distributions in  $I_{t_0}$  and  $\lim_{\ell \rightarrow \infty} b^\ell = b$  in  $(\mathcal{D}'(\mathbf{R}))^n$ , then  $x_0$  is consistent with all the DAE's obtained by replacing  $b$  by  $b^\ell$  in (3.17\_-) (resp. (3.17\_+)), and denoting by  $x^\ell \in C_{\text{imp}}^n(\mathbf{R}^*)$  the unique solution of the initial value problem*

$$(3.19) \quad A\dot{x}^\ell + Bx^\ell = b^\ell, \quad x^\ell(t_0) = x_0,$$

*we have*

$$(3.20) \quad \lim_{\ell \rightarrow \infty} x^\ell = x \quad \text{in } (\mathcal{D}'(\mathbf{R}))^n.$$

(ii) If  $t_0 = 0$  and  $x_0 \in \mathbb{R}^n$  is consistent with the DAE (3.17<sub>-</sub>) (resp. (3.17<sub>+</sub>)) at  $t_0 = 0$ , the initial value problem

$$(3.21_{\pm}) \quad A\dot{x} + Bx = b, \quad x_{-}(0) = x_0 \text{ (resp. } x_{+}(0) = x_0),$$

has a unique solution  $x \in C_{\text{imp}}^n(\mathbb{R}^*)$ . Furthermore, if  $I_0$  is an open interval about 0 and  $b^\ell \in C_{\text{imp}}^n(\mathbb{R}^*)$  is a sequence such that  $b_{|I_0}^\ell = b_{|I_0}$ , for all  $\ell \geq 1$ , as distributions in  $I_0$  and  $\lim_{\ell \rightarrow \infty} b^\ell = b$  in  $(\mathcal{D}'(\mathbb{R}))^n$ , then  $x_0$  is consistent with all the DAE's obtained by replacing  $b$  by  $b^\ell$  in (3.17<sub>-</sub>) (resp. (3.17<sub>+</sub>)) and denoting by  $x^\ell \in C_{\text{imp}}^n(\mathbb{R}^*)$  the unique solution of the initial value problem

$$(3.22) \quad A\dot{x}^\ell + Bx^\ell = b^\ell, \quad x_{-}^\ell(0) = x_0 \quad (\text{resp. } x_{+}^\ell(0) = x_0),$$

we have

$$(3.23) \quad \lim_{\ell \rightarrow \infty} x^\ell = x \quad \text{in } (\mathcal{D}'(\mathbb{R}))^n.$$

*Proof:* In this proof, we use the notation of Section 2 without further mention. From the reduction procedure we know that every solution  $x \in (\mathcal{D}'(\mathbb{R}))^n$  of the DAE (3.16) has the form  $x = \Gamma_{\nu-1}x_\nu + v_{\nu-1}$  where  $\Gamma_{\nu-1}$  and  $v_{\nu-1}$  are given by (2.6) and (2.7), respectively, and  $x_\nu$  solves the ODE

$$(3.24) \quad \dot{x}_\nu + A_\nu^{-1}B_\nu x_\nu = A_\nu^{-1}b_\nu.$$

The key point here is the simple fact that the distributions  $u_j$  and  $b_j$  of Section 2 belong to  $C_{\text{imp}}^{r_{j-1}}(\mathbb{R}^*)$  and have impulse order at most  $k+j$ . Indeed, recall that  $b_0 = b$  and  $r_{-1} = n$ , whence, because of  $u_0 = B^T(AA^T + BB^T)^{-1}b$ , we have  $u_0 \in C_{\text{imp}}^n(\mathbb{R}^*)$  and  $\text{iord}(u_0) \leq \text{iord}(b) = k$  by Remark 3.2. Therefore,  $u_0 \in C_{\text{imp}}^n(\mathbb{R}^*)$  has impulse order at most  $k+1$  which, in turn, implies that  $b_1 = D(b - Bu_0 - Au_0) \in C_{\text{imp}}^{r_0}(\mathbb{R}^*)$  has impulse order at most  $k+1$ . Obviously, the statement about the sequences  $u_0, \dots, u_{\nu-1}, b_0, \dots, b_\nu$  now follows inductively by the same argument.

Since  $b_\nu \in C_{\text{imp}}^{r_\nu-1}(\mathbb{R}^*)$  has impulse order at most  $k + \nu - 1$ , the same is true of  $A_\nu^{-1}b_\nu$ . Therefore, by Lemma 3.1, the solutions of (3.24) are in  $C_{\text{imp}}^{r_\nu-1}(\mathbb{R}^*)$  and have impulse order at most  $k + \nu - 1$ . This implies that the solutions  $x = \Gamma_{\nu-1}x_\nu + v_\nu$  of (3.16) are in  $C_{\text{imp}}^n(\mathbb{R}^*)$ . Moreover,  $\text{iord}(\Gamma_{\nu-1}x_\nu) \leq k + \nu - 1$  since  $\Gamma_{\nu-1}$  is  $C^\infty$ , and  $\text{iord}(v_\nu) \leq k + \nu - 1$ , because the  $C_j$ 's are  $C^\infty$  and  $\text{iord}(u_j) \leq k + j$  for  $0 \leq j \leq \nu - 1$ . Thus  $x$  has impulse order at most  $k + \nu - 1$ .

If now  $t_0 < 0$  (resp.  $t_0 > 0$ ) and  $x_0 \in \mathbb{R}^n$  is consistent with the DAE (3.17<sub>-</sub>) (resp. (3.17<sub>+</sub>)), there is a unique  $x_{0\nu} \in \mathbb{R}^{r_\nu-1}$  such that  $x_0 = \Gamma_{\nu-1}(t_0)x_{0\nu} + v_{\nu-1}(t_0)$ , (note that  $v_{\nu-1}(t_0)$  makes sense since  $t_0 \neq 0$ ). Hence, the solution  $x$  of (3.18) is obtained as  $x = \Gamma_{\nu-1}x_\nu + v_{\nu-1}$  where, in line with Theorem 3.1,  $x_\nu \in C_{\text{imp}}^{r_\nu-1}(\mathbb{R}^*)$  is the unique solution of

$$(3.25) \quad \dot{x}_\nu + A_\nu^{-1}B_\nu x_\nu = b_\nu, \quad x_\nu(t_0) = x_{0\nu},$$

and no other initial values can be substituted for  $x_{0\nu}$  because  $\Gamma_{\nu-1}(t_0)$  is one-to-one.

For the "furthermore" part in (i) of the theorem, observe first that consistency of a value  $x_0 \in \mathbb{R}^n$  with a (linear) DAE at a point  $t_0$  depends only upon the coefficients and the right-hand side of the DAE in an arbitrarily small neighborhood of  $t_0$ . As a result, the hypothesis  $b'|_{t_0} = b|_{t_0}$  ensures that  $x_0$  remains consistent with the DAE obtained by replacing  $b$  by  $b'$  in (3.17<sub>-</sub>) (resp. (3.17<sub>+</sub>)).

For fixed  $\ell \geq 1$ , denote by  $u_j^\ell, 0 \leq j \leq \nu - 1$  and  $b_j^\ell, 0 \leq j \leq \nu$ , the sequences corresponding to  $u_j, b_j$  in the procedure of Section 2 after replacing  $b$  by  $b'$ , and let  $v_{\nu-1}^\ell$  be defined by (2.7) with  $u_0, \dots, u_{\nu-1}$  replaced by  $u_0^\ell, \dots, u_{\nu-1}^\ell$ , respectively. With  $\Gamma_{\nu-1}$  as in (2.6), we find that the solution  $x^\ell$  of (3.19) has the form

$$(3.26) \quad x^\ell = \Gamma_{\nu-1}x_\nu^\ell + v_{\nu-1}^\ell,$$

where  $x_\nu^\ell \in C_{\text{imp}}^{r_\nu-1}(\mathbb{R}^*)$  solves the initial value problem

$$(3.27) \quad \dot{x}_\nu^\ell + A_\nu^{-1}B_\nu x_\nu^\ell = A_\nu^{-1}b_\nu^\ell, \quad x_\nu^\ell(t_0) = x_0.$$

From the hypothesis  $b_{j|t_0}^\ell = b_{j|t_0}$ , it follows at once that  $u_{j|t_0}^\ell = u_{j|t_0}$  and  $b_{j|t_0}^\ell = b_{j|t_0}$  for all the indices  $\ell, j$  of interest. In particular,  $b_{\nu|t_0}^\ell = b_{\nu|t_0}$  and hence

$$(3.28) \quad A_\nu^{-1} b_{\nu|t_0}^\ell = A_\nu^{-1} b_{\nu|t_0}, \quad \forall \ell \geq 1.$$

Next, the hypothesis  $\lim_{\ell \rightarrow \infty} b^\ell = b$  in  $(\mathcal{D}'(\mathbb{R}))^n$  and continuity of the multiplication of distributions by  $C^\infty$  matrix-valued functions yield  $\lim_{\ell \rightarrow \infty} u_j^\ell = u_j$  in  $(\mathcal{D}'(\mathbb{R}))^{r_{j-1}}$ ,  $0 \leq j \leq \nu-1$  and  $\lim_{\ell \rightarrow \infty} b_j^\ell = b_j$  in  $(\mathcal{D}'(\mathbb{R}))^{r_{j-1}}$ ,  $0 \leq j \leq \nu$ . In particular,

$$(3.29) \quad \lim_{\ell \rightarrow \infty} A_\nu^{-1} b_\nu^\ell = A_\nu^{-1} b_\nu \quad \text{in } (\mathcal{D}'(\mathbb{R}))^{r_{\nu-1}}$$

and

$$(3.30) \quad \lim_{\ell \rightarrow \infty} v_{\nu-1}^\ell = v_{\nu-1} \quad \text{in } (\mathcal{D}'(\mathbb{R}))^n.$$

Since  $x_\nu$  and  $x_\nu^\ell$  solve the initial value problems (3.25) and (3.27), respectively, it follows from (3.28) and (3.29) and Theorem 3.1 (iii) that  $\lim_{\ell \rightarrow \infty} x_\nu^\ell = x_\nu$  in  $(\mathcal{D}'(\mathbb{R}))^{r_{\nu-1}}$ . Together with (3.26) and (3.30), this implies that  $\lim_{\ell \rightarrow \infty} x^\ell = \Gamma_{\nu-1} x_\nu + v_{\nu-1} = x$  in  $(\mathcal{D}'(\mathbb{R}))^n$ . This completes the proof of part (i) of the theorem.

Finally, for the proof of (ii), if  $t_0 = 0$  and  $x_0 \in \mathbb{R}^n$  is consistent with the DAE (3.17<sub>-</sub>) (resp. (3.17<sub>+</sub>)) at  $t_0 = 0$ , then the solution  $x$  of (3.21<sub>±</sub>) is obtained in the form  $x = \Gamma_{\nu-1} x_\nu + v_{\nu-1}$  where, by Theorem 3.1,  $x_\nu$  is the unique solution of

$$\dot{x}_\nu + A_\nu^{-1} B_\nu x_\nu = b_\nu, \quad x_{\nu-}(0) = x_{\nu 0} \quad (\text{resp. } x_{\nu+}(0) = x_{\nu 0}),$$

and  $x_{\nu 0} \in \mathbb{R}^{r_{\nu-1}}$  is (by injectivity of  $\Gamma_{\nu-1}(0)$ ) the unique solution of the equation  $x_0 = \Gamma_{\nu-1}(0)x_{\nu 0} + v_{\nu-}(0)$  (resp.  $x_0 = \Gamma_{\nu-1}(0)x_{\nu 0} + v_{\nu+}(0)$ ). The proof of the remaining statement is identical to the proof of the "furthermore" part in (i) of the theorem.  $\square$

**Remark 3.5:** Let "[ ]" stand for "jump at 0". Then, since every solution  $x$  of the DAE (3.16) has the form  $x = \Gamma_{\nu-1} x_\nu + v_{\nu-1}$  with  $x_\nu$  solving the ODE (3.24), we have

$[x] = \Gamma_{\nu-1}(0)[x_\nu] + [v_{\nu-1}]$  and  $x_{\text{imp}} = (\Gamma_{\nu-1}x_\nu)_{\text{imp}} + v_{\nu \text{ imp}}$ . On the other hand,  $\Gamma_{\nu-1}$  and  $v_{\nu-1}$  are obtained through an explicit procedure. As a result,  $[x]$  and  $x_{\text{imp}}$  can be calculated if  $[x_\nu]$  and  $x_{\nu \text{ imp}}$  are known (using (3.3) - (3.4) for the term  $(\Gamma_{\nu-1}x_\nu)_{\text{imp}}$ ). But from Remark 3.4,  $[x_\nu]$  and  $x_{\nu \text{ imp}}$  can be evaluated from  $b_{\nu \text{ imp}}$ , and  $b_{\nu \text{ imp}}$  is calculable since  $b_\nu$  is known explicitly. Thus, both  $[x]$  and  $x_{\text{imp}}$  are calculable, at least in principle.  $\square$

#### 4. Inconsistent initial values.

Let  $A, B \in C^\infty(\mathbb{R}; \mathcal{L}(\mathbb{R}^n))$  and  $b_+ \in C^\infty([0, \infty); \mathbb{R}^n)$  be given. As noted in the Introduction, the problem of solving the initial value problem

$$(4.1) \quad A(t)\dot{x} + B(t)x = b_+(t), \quad \text{in } (0, \infty),$$

$$(4.2) \quad x(0) = x_0,$$

for arbitrary  $x_0 \in \mathbb{R}^n$  that is not necessarily consistent with the DAE (4.1) at  $t_0 = 0$ , often arises when a known function  $x_-$  on  $(-\infty, 0]$  verifies  $x_-(0) = x_0$  and is to be extended into a solution of the DAE in (4.1). A general approach is suggested by the following observation:

**Lemma 4.1.** *Let  $x_- \in C^\infty((-\infty, 0]; \mathbb{R}^n)$  be given, and suppose that  $x_0 = x_-(0)$  is consistent with the DAE (4.1) at  $t_0 = 0$ . Assume further that the pair  $(A, B)$  has index  $\nu \geq 0$  in  $\mathbb{R}$  and that, in the notation of Section 2,  $A_\nu(t)$  is invertible for every  $t \in \mathbb{R}$ . Let  $x_+ \in C^\infty([0, \infty); \mathbb{R}^n)$  be the unique solution of (4.1) and (4.2). Set*

$$(4.3) \quad b(t) = \begin{cases} A(t)\dot{x}_-(t) + B(t)x_-(t) & \text{if } t < 0, \\ b_+(t) & \text{if } t > 0, \end{cases}$$

so that  $b \in C_{\text{imp}}^n(\mathbb{R}^*)$  (and  $b_{\text{imp}} = 0$ ). Then, the function

$$(4.4) \quad x(t) = \begin{cases} x_-(t) & \text{if } t < 0, \\ x_+(t) & \text{if } t > 0, \end{cases}$$

verifies  $x \in C_{\text{imp}}^n(\mathbb{R}^*) \cap (C^0(\mathbb{R}))^n$  and is the unique solution of both initial value problems

$$(4.5) \quad A\dot{x} + Bx = b \quad \text{in } \mathbb{R}, \quad x_-(0) = x_0,$$

$$(4.6) \quad A\dot{x} + Bx = b \quad \text{in } \mathbb{R}, \quad x_+(0) = x_0.$$

*Proof:* It is obvious that  $x \in C_{\text{imp}}^n(\mathbb{R}^*)$ . In particular, the derivative  $\dot{x} \in (\mathcal{D}'(\mathbb{R}))^n$  is the function given by  $\dot{x}_-(t)$  for  $t < 0$  and by  $\dot{x}_+(t)$  for  $t > 0$ , whence  $A\dot{x} + Bx = b$  in  $\mathbb{R}$  in the sense of distributions. By definition of  $x_-$  and  $b$ ,  $x_0 = x_-(0)$  is consistent with the DAE (3.17<sub>-</sub>) at  $t_0 = 0$ , and by hypothesis  $x_0$  is also consistent with the DAE (3.17<sub>+</sub>) at  $t_0 = 0$ . It then follows from Theorem 3.2 that (4.5) and (4.6) each have a unique solution in  $C_{\text{imp}}^n(\mathbb{R}^*)$ . In both cases, this solution is  $x$  since  $x_-(0) = x_+(0) = x_0$  by continuity of  $x$  at 0.  $\square$

Lemma 4.1 suggests that we should solve (4.1) for inconsistent  $x_0$  by making use of the extension  $b$  of  $b_+$  in (4.3). This approach is taken in the following result:

**Theorem 4.1.** Let  $x_- \in C^\infty((-\infty, 0]; \mathbb{R}^n)$  and  $b_+ \in C^\infty([0, \infty); \mathbb{R}^n)$  be given. Suppose that the pair  $(A, B)$  has index  $\nu \geq 0$  and that, in the notation of Section 2,  $A_\nu(t)$  is invertible for every  $t \in \mathbb{R}$ . Then, for  $b \in C_{\text{imp}}^n(\mathbb{R}^*)$  defined by (4.3), there exists a unique distribution  $x \in (\mathcal{D}'(\mathbb{R}))^n$  which solves

$$(4.7) \quad A\dot{x} + Bx = b \quad \text{in } \mathbb{R}, \quad x|_{(-\infty, 0)} = x_-.$$

Moreover, we have

(i)  $x \in C_{\text{imp}}^n(\mathbb{R}^*)$  and  $x$  has impulse order at most  $\nu - 2$ .

(ii)  $x_+ = x|_{(0, \infty)}$  solves the DAE

$$(4.8) \quad A(t)\dot{x}_+ + B(t)x_+ = b_+(t) \quad \text{in } (0, \infty).$$

(iii) If  $x_0 \equiv x_-(0)$  is consistent with the DAE (4.8) at  $t_0 = 0$ , then  $x \in (C^0(\mathbb{R}))^n$  and  $x_+$  is the classical solution of the initial value problem

$$(4.9) \quad A(t)\dot{x}_+ + B(t)x_+ = b_+(t) \quad \text{in } (0, \infty), \quad x_+(0) = x_0.$$

(iv) Irrespective of the consistency of  $x_0 \equiv x_-(0)$  with the DAE (4.8) at  $t_0 = 0$ , the distribution  $\xi = x_+ + x_{\text{imp}}$ , with  $x_+$  extended by 0 in  $(-\infty, 0)$ , is the unique solution in  $C_{\text{imp}}^n(\mathbb{R}^*)$  of the initial value problem

$$(4.10) \quad A\dot{\xi} + B\xi = b_+ + A(0)x_0\delta \quad \text{in } \mathbb{R}, \quad \xi_-(0) = 0,$$

where  $b_+$  is extended by 0 in  $(-\infty, 0)$ . In particular,  $\text{iord}(\xi) \leq \max(-1, \nu - 2)$  and hence  $\text{iord}(\xi) \leq \nu - 2$  for  $\nu \geq 1$  (in contrast to  $\text{iord}(\xi) \leq \nu - 1$  obtained by a direct application of Theorem 3.2 to (4.10)).

*Proof:* Set  $x_0 = x_-(0)$  and  $b_-(t) = A(t)\dot{x}_-(t) + B(t)x_-(t)$  for  $t \leq 0$ , so that by definition  $x_-$  solves the DAE (3.17<sub>-</sub>) and  $x_0$  is consistent with (3.17<sub>-</sub>) at  $t_0 = 0$ . Thus, by Theorem 3.2 there is a unique solution  $y \in C_{\text{imp}}^n(\mathbb{R}^*)$  of the initial value problem

$$(4.11) \quad A\dot{y} + By = b \quad \text{in } \mathbb{R}, \quad y_-(0) = x_0.$$

As a result,  $y_-$  is another solution of (3.17<sub>-</sub>) which, just as  $x_-$ , verifies  $y_-(0) = x_0$ . This implies that  $y_- = x_-$  and hence that  $x = y$  solves (4.7). Conversely, if  $y \in (\mathcal{D}'(\mathbb{R}))^n$  solves (4.7), then  $y \in C_{\text{imp}}^n(\mathbb{R}^*)$  by Theorem 3.2, and the equality  $y|_{(-\infty, 0)} = x_-$  as distributions in  $(-\infty, 0)$  implies at once that  $y_- = x_-$ . Thus, by continuity,  $y_-(0) = x_-(0) = x_0$ , and  $y$  solves (4.11). Uniqueness of the solution of (4.7) then follows from the unique solvability of (4.11).

We now pass to the proof of the statements (i) - (iv). Part (i) follows from Theorem 3.2 and the fact that the right side  $b$  in (4.7) and (4.11) (which, as was just seen, have the same solution) is given by (4.3), and hence has impulse order  $k \leq -1$ . Property (ii) is a trivial consequence of (i) and the fact that  $x$  solves (4.7). For the proof of (iii) note that if  $x_0 \equiv x_-(0)$  is also consistent with the DAE (4.8) at  $t_0 = 0$ , then Theorem 3.1 ensures that the unique solution  $y = x$  of (4.11) (and hence also of (4.7)) is given by (4.4) and solves (4.6). This shows that  $x \in (C^0(\mathbb{R}))^n$  and that  $x_+$  solves (4.9).

For the proof of (iv), set  $\xi = x_+ + x_{\text{imp}} \in C_{\text{imp}}^n(\mathbb{R}^*)$ , so that  $\xi = x - x_-$  with  $x_-$

extended by 0 in  $(0, \infty)$ . Since  $\text{iord}(x) \leq \nu - 2$  and  $\text{iord}(x_-) \leq -1$ , we find that  $\text{iord}(\xi) \leq \max(-1, \nu - 2)$ . Moreover, viewing  $x_-$  as a function of  $t \in \mathbb{R}$ , we have  $A\dot{x}_- + Bx_- = b_- - Ax_0\delta$ , where

$$b_-(t) = \begin{cases} A(t)\dot{x}_-(t) + B(t)x_-(t) & \text{if } t < 0, \\ 0 & \text{if } t > 0. \end{cases}$$

From the above discussion and the definition of  $b$  in (4.3), it follows at once that  $A\dot{\xi} + B\xi = A\dot{x} + Bx - A\dot{x}_- - Bx_- = b_+ + Ax_0\delta$ . Moreover, we have  $Ax_0\delta = A(0)x_0\delta$  and  $\xi_-(0) = x_-(0) - x_-(0) = 0$  whence  $\xi$  solves (4.10) and thus coincides with the unique solution of that problem. Note here that the consistency of  $0 \in \mathbb{R}^n$  with the DAE  $A(t)\dot{\xi}_- + B(t)\xi_- = (b_+)_-(t) = 0$  in  $(-\infty, 0)$  follows from the fact that  $\xi_-(t) = 0$  is a solution.  $\square$

Theorem 4.1 justifies the choice of the solution  $\xi$  of (4.10) (or of its positive part  $\xi_+$ ) to represent the solution  $x$  of (4.1) when  $x_0$  is not consistent with (4.1) at  $t_0 = 0$ . Further justification will be provided by Theorem 4.2 below. For the time being observe that the characterization (4.10) of  $\xi = x_+ + x_{\text{imp}}$  shows that the extension  $x$  of  $x_-$  as a solution of (4.7) depends only upon  $x_0 \equiv x_-(0)$  and  $b_+$  and hence is independent of  $x_-(t)$  for  $t < 0$ . For the case when  $A$  and  $B$  are constant, the characterization (4.10) is exactly that of [VLK81], [Co82], [G93], but the equivalent characterization (4.7) is not explicitly noticed in these papers.

It is noteworthy that for index 1 problems,  $\xi$  solving (4.10) has impulse order at most  $-1$ , and hence  $\xi_{\text{imp}} = 0$ . In other words,  $\xi$  is a function with a possible discontinuity at the origin. A simple formula can be given for the jump  $\xi_+(0)$  of  $\xi$  (recall  $\xi_- = 0$ ) without using the more cumbersome general procedure outlined in Remark 3.5. Indeed, the relation  $A\dot{\xi} + B\xi = b_+ + A(0)x_0\delta$  reads  $A\xi_+(0)\delta + A(d\xi/dt) + B\xi = b_+ + A(0)x_0\delta$ , where  $d\xi/dt$  denotes the usual derivative of  $\xi$  at points of  $\mathbb{R}^*$ . Clearly, this requires that  $A(0)\xi_+(0) = A(0)x_0$  and that  $A(t)(d\xi/dt)(t) + B(t)\xi(t) = b_+(t)$ ,  $\forall t \in \mathbb{R}^*$ . In particular, for  $t > 0$ , we must have  $Q_0(t)B(t)\xi_+(t) = Q_0(t)b_+(t)$  where  $Q_0 \in C^\infty(\mathbb{R}; \mathcal{L}(\mathbb{R}^n))$  is as in Section 2; that is,  $Q_0(t)$  projects onto a complement of  $\text{rge } A(t)$ , (or of  $\text{ext rge } A(t)$  in the analytic case). By continuity, we obtain  $Q_0(0)B(0)\xi_+(0) = Q_0(0)b_+(0)$ . Thus,  $\xi_+(0)$



solves the system

$$(4.12) \quad (A(0) + Q_0(0)B(0))\xi_+(0) = A(0)x_0 + Q_0(0)b_+(0).$$

Conversely, if  $\xi_+(0)$  solves (4.12) then  $A(0)\xi_+(0) = A(0)x_0$  and because of  $\text{rge } A(0) \cap \text{rge } Q_0(0) = \{0\}$  we have  $Q_0(0)B(0)\xi_+(0) = Q_0(0)b_+(0)$ . It turns out (see [RR93a]) that invertibility of  $A(t) + Q_0(t)B(t)$  for every  $t \in \mathbb{R}$  is implied by the index 1 assumption, and hence that  $\xi_+(0)$  is given by

$$\xi_+(0) = [A(0) + Q_0(0)B(0)]^{-1}(A(0)x_0 + Q_0(0)b_+(0)).$$

Note that  $\xi_+(0) = x_0$  if and only if  $Q_0(0)B_+(0)x_0 = Q_0(0)b_+(0)$  (see (4.12)), a condition that is easily seen to be equivalent to the consistency of  $x_0$  with the DAE (3.17<sub>+</sub>) at  $t_0 = 0$ .

In Theorem 4.1, the function  $b(t)$  may be approximated, in the sense of  $(\mathcal{D}'(\mathbb{R}))^n$ , by sequences of smooth functions  $b^\ell \in C^\infty(\mathbb{R}; \mathbb{R}^n)$ . In practice, considering such a sequence amounts to viewing the transition from  $x_-$  to  $x_+$  as the limiting case of a perhaps physically more realistic, situation where a rapid but not discontinuous modification of the input occurs in the vicinity of  $t = 0$ . In this setting, it is perfectly reasonable to assume that  $b^\ell = b_-$ , for all  $\ell \geq 1$ , in some interval  $(-\infty, -a]$  for some  $a > 0$  independent of  $\ell$ . On the other hand, the function  $x_{-|(-\infty, -a]}$  has a unique extension as a solution  $x^\ell \in C^\infty(\mathbb{R}; \mathbb{R}^n)$  of the DAE

$$Ax^\ell + Bx^\ell = b_\ell \quad \text{in } \mathbb{R}.$$

In fact,  $x^\ell$  can be obtained as the solution of the initial value problem

$$(4.13) \quad Ax^\ell + Bx^\ell = b_\ell \quad \text{in } \mathbb{R}, \quad x^\ell(t_0) = x_-(t_0),$$

where  $t_0 \leq -a$  is arbitrarily chosen. Evidently, it would be desirable that the sequence  $x^\ell$  tends to the solution  $x$  of (4.7) in some sense. That this is indeed true, and more specifically that  $\lim_{\ell \rightarrow \infty} x^\ell = x$  in  $(\mathcal{D}'(\mathbb{R}))^n$  follows at once from Theorem 3.2 and the hypotheses  $b^\ell = b$  in  $(-\infty, -a]$ , for all  $\ell \geq 1$ , and  $\lim_{\ell \rightarrow \infty} b^\ell = b$  in  $(\mathcal{D}'(\mathbb{R}))^n$  (just choose  $t_0 < -a$  in (4.13)). We record this result in the following form:

**Theorem 4.2.** Let  $x_- \in C^\infty((-\infty, 0], \mathbf{R})$  and  $b_+ \in C^\infty([0, \infty); \mathbf{R}^n)$  be given. Suppose that the pair  $(A, B)$  has index  $\nu \geq 0$  and that, in the notation of Section 2,  $A_\nu(t)$  is invertible for  $t \in \mathbf{R}$ . Let  $b \in C_{\text{imp}}^n(\mathbf{R}^*)$  be defined by (4.3) and let  $b^\ell \in C^\infty(\mathbf{R}; \mathbf{R}^n)$ ,  $\ell \geq 1$ , be a sequence such that  $b^\ell = b_-$  in  $(-\infty, -a]$  for some  $a \geq 0$  independent of  $\ell$  and such that  $\lim_{\ell \rightarrow \infty} b^\ell = b$  in  $(\mathcal{D}'(\mathbf{R}))^n$ . Denote by  $x^\ell \in C^\infty(\mathbf{R}; \mathbf{R}^n)$  the unique extension of  $x_-|_{(-\infty, -a]}$  as a solution of the DAE

$$Ax^\ell + Bx^\ell = b^\ell \quad \text{in } \mathbf{R},$$

and let  $x \in C_{\text{imp}}^n(\mathbf{R}^*)$  be the solution of (4.7). Then, we have

$$\lim_{\ell \rightarrow \infty} x^\ell = x \quad \text{in } (\mathcal{D}'(\mathbf{R}))^n.$$

## 5. Some generalizations.

Let  $\mathcal{J} \subset \mathbf{R}$  be an open interval and let  $\mathcal{S} = (a_i)_{i \in \mathbf{Z}}$  be a nondecreasing sequence of points of  $\mathbf{R} \cup \{\pm\infty\}$  with  $a_i < a_{i+1}$  if either  $a_i$  or  $a_{i+1}$  is real and  $\lim_{i \rightarrow \pm\infty} a_i \notin \mathcal{J}$ . Denote by  $C_{\text{imp}}(\mathcal{J} \setminus \mathcal{S})$  the subspace of  $\mathcal{D}'(\mathcal{J})$  of the distributions of the form  $x = \tilde{x} + x_{\text{imp}}$  where  $\tilde{x}$  is a function such that  $\tilde{x}|_{(a_i, a_{i+1}) \cap \mathcal{J}} \in C^\infty([a_i, a_{i+1}] \cap \bar{\mathcal{J}})$ ,  $\forall i \in \mathbf{Z}$ , and  $x_{\text{imp}}$  is a distribution with support contained in  $\mathcal{S} \cap \mathcal{J}$ . Equivalently, if  $\delta_{a_i}$  is the Dirac delta distribution at  $a_i$ , then  $x_{\text{imp}}$  is a finite or infinite linear combination of derivatives of  $\delta_{a_i}$  with  $a_i \in \mathcal{J}$ . With this definition of  $C_{\text{imp}}(\mathcal{J} \setminus \mathcal{S})$ , we have  $C_{\text{imp}}(\mathcal{J}) = C^\infty(\bar{\mathcal{J}})$  if  $\mathcal{J} \cap \mathcal{S} = \emptyset$ .

With the definition  $C_{\text{imp}}^n(\mathcal{J} \setminus \mathcal{S}) = (C_{\text{imp}}(\mathcal{J} \setminus \mathcal{S}))^n$ , it should be evident how to formulate Theorems 3.1 and 3.2 for the case  $b \in C_{\text{imp}}^n(\mathcal{J} \setminus \mathcal{S})$ . Of course, the elements of  $C_{\text{imp}}^n(\mathcal{J} \setminus \mathcal{S})$  have an impulse order at each point  $a_i \in \mathcal{J} \cap \mathcal{S}$ . It may only be useful to note that for a given arbitrary sequence  $\mu_{oi} \in \mathbf{R}^n$ , a primitive of  $\sum_{i=-\infty}^{\infty} \mu_{oi} \delta_{a_i}$  is  $\sum_{i=-\infty}^{-1} (-\mu_{oi})(1 - H_{a_i}) + \sum_{i=0}^{\infty} \mu_{oi} H_{a_i}$ , where  $H_{a_i}(t) = H(t - a_i)$  if  $a_i \in \mathbf{R}$ ,  $H_{-\infty}(t) = 1$ ,  $H_{\infty}(t) = 0$ , and not  $\sum_{i=-\infty}^{\infty} \mu_{oi} H_{a_i}$  which would not make sense when  $\sum_{i=-\infty}^{-1} \mu_{oi}$  does not converge.

For  $0 \in \mathcal{J}$  it should be equally obvious how solutions of the initial value problem

$$Ax + Bx = b_+ \quad \text{in } \mathcal{J}_+ = \mathcal{J} \cap (0, \infty), \quad x(0) = x_0,$$

can be defined when  $0 \notin S$ ,  $b_+ \in C_{\text{imp}}^n(\mathcal{J}_+ \setminus S)$ , and  $x_0 \in \mathbb{R}^n$  is not consistent with the DAE  $A\dot{x} + Bx = b_+$  in  $\mathcal{J}_+$  at  $t = 0$ . For problems with index  $\nu \geq 2$  the solutions may exhibit a nonzero impulse at  $t_0 = 0$ . The case considered in Section 4 corresponds to  $\mathcal{J} = \mathbb{R}$ ,  $S = \{\infty\}$ .

## 6. Numerical examples.

For index-one problems the computation of the jump (4.12) caused by inconsistent input can be easily incorporated into a numerical procedure for solving the initial value problems (4.1/2). We consider here a recently developed solution process, [RR93b], which is based on the reduction procedure of [RR93a] summarized in Section 2 above.

Suppose that the DAE (4.1) has index 1. For a given step  $h > 0$  set  $t_i = ih$ ,  $i = 0, 1, \dots$ , and consider the explicit Euler approximation

$$(6.1) \quad A(t_i) \frac{1}{h}(x_{i+1} - x_i) + B(t_i)x_i = b_+(t_i).$$

In [RR93b], (Theorem 3), it was shown that any solution  $x_0, x_1, \dots, x_m \in \mathbb{R}^n$  verifies for  $i = 0, 1, \dots, m-1$  the equations  $Q(t_i)B(t_i)x_i = Q(t_i)b_+(t_i)$  and

$$(6.2) \quad [A(t_i) + Q(t_{i+1})B(t_{i+1})]x_{i+1} = [A(t_i) - hB(t_i)]x_i + hb_+(t_i) + Q(t_{i+1})b_+(t_{i+1}).$$

Conversely, for sufficiently small  $h$  and any given  $x_0 \in \mathbb{R}^n$  such that  $Q(0)B(0)x_0 = Q(0)b_+(0)$ , the solution  $x_0, x_1, \dots, x_m$  of (6.2) is unique and solves also (6.1). Smallness of  $h$  ensures that the operator  $A(t_i) + Q(t_{i+1})B(t_{i+1})$  is invertible, given that invertibility of  $A(t) + Q(t)B(t)$  for all  $t$  is equivalent to the index 1 assumption.

The difference scheme (6.2) has been used as the base method in an explicit extrapolation integrator, LTV1XE, for general index-one problems (4.1) - (4.2).

Now note that for  $t_0 = 0$ ,  $h = 0$ , and with  $x_{i+1}$  replaced by  $\xi_+(0)$  the difference equation (6.2) is identical with (4.12). Thus, the results of Section 4 ensure that for any given  $x_0$  we only need to apply (6.2) with  $h = 0$  to obtain the consistent starting point from which

the solution process can then be started. This represents only a minor modification to the mentioned code LTV1XE. The resulting code accepts any given initial point and then computes the solution starting from the corresponding solution of (4.12).

As an example consider the index-one problem

$$(6.3) \quad \begin{pmatrix} 1 & -t & t^2 \\ 0 & 1 & -t \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{pmatrix} + \begin{pmatrix} 1 & -(t+1) & t^2 + 2t \\ 0 & -1 & t-1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \sin t \end{pmatrix}$$

given in [CP88] which has the general solution

$$(6.4) \quad x(t) = (\alpha t e^t + \beta e^{-t}, \alpha e^t + t \sin t, \sin t)^T \in \mathbb{R}^3.$$

For several randomly selected points  $(x_1, x_2, x_3)^T$  and starting times, Table 6.1 gives the corresponding consistent starting points computed by LTV1XE. It is readily checked that these consistent points verify (6.4) for suitable constants  $\alpha$  and  $\beta$ .

point-type	$t$	$x_1$	$x_2$	$x_3$
given pnt.	0.0	0.0	1.0	1.0
consistent	0.0	0.0	1.0	0.0
given pnt.	1.0	1.0	1.0	1.0
consistent	1.0	1.0	0.84147098	0.84147098
given pnt.	1.0	-1.0	4.0	5.0
consistent	1.0	-1.0	-0.15852902	0.84147098
given pnt.	2.0	-5.0	2.0	3.0
consistent	2.0	-5.0	-2.1814051	0.90929743
given pnt.	-1.0	1.0	-1.0	2.0
consistent	-1.0	1.0	1.8414710	-0.84147098

Table 6.1: Consistent points for (6.3)

For the index-two case a code LTV2XE was developed which incorporates the reduction discussed in Section 2 for a given index-two problem (4.1) and then applies LTV1XE to the reduced index-one problem. The central part in the reduction is the computation of the mappings  $C$  and  $D$ . This can be implemented, in general, by using a singular value

decomposition (SVD) to obtain a basis of  $\text{rge } A$  and the projection  $Q$ , and another SVD for generating a basis of  $\ker QB$ . But, it turns out that in many cases there are much simpler ways of generating these mappings. Thus LTV2XE assumes that subroutines are available not only for the coefficients  $A$ ,  $B$ ,  $b$  and their derivatives, but also for  $C$  and  $D$ . This allows us to bypass easily the costly general method for calculating these matrices whenever a simpler approach is feasible.

In this form LTV2XE will work as long as the coefficients of the problem are smooth. When the right side of the original equation has a jump, then, in general, the right side of the reduced equation exhibits not only a jump but also an impulse. Hence the earlier given simple jump computation (4.12) for index one problems is insufficient for the index-two case.

As an illustration consider the simple DAE

$$(6.5) \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \dot{x} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} x = \begin{pmatrix} 1 \\ t \\ \tau(t) \end{pmatrix}$$

with the initial condition  $x^0 = (0, 0, 1)^T$ . When  $\tau$  is a smooth function with  $\tau(0) = 1$ ,  $\dot{\tau}(0) = 0$  then the unique solution is

$$(6.6) \quad x_1(t) = \tau(t), \quad x_2(t) = t - 1 + \exp(-t), \quad x_3(t) = 1 - \dot{\tau}(t).$$

Suppose now that  $\tau(t) = H_1(t)$  where  $H_1$  is the Heaviside function with the step at  $t = 1$ . Then the solution has the same form as (6.6) but with  $x_1(t) = H_1(t)$  and  $x_3(t) = 1 - \delta_1(t)$ . Thus at  $t = 1$  we have a jump of size 1 in the first component and an impulse of size  $-1$  in the third component. A graph of this solution does not show the impulse. But if we approximate the step of  $H_1$  by a cubic spline; that is, if we consider (6.5) with

$$(6.7) \quad \tau(t) = \begin{cases} 0, & \text{for } 0 \leq t < 1 - \epsilon, \\ \frac{1}{2} + \frac{1}{4} \sigma(t)(3 - \sigma(t)^2), & \text{for } 1 - \epsilon \leq t \leq 1 + \epsilon, \quad \sigma(t) = \frac{t-1}{\epsilon}, \\ 1, & \text{for } t > 1 + \epsilon, \end{cases}$$

with small  $\epsilon > 0$ , then (6.6) shows that  $x_3(1) = 1 - 3/(4\epsilon)$ . In other words, this solution approximates the impulse.

For the general computation of the jump and impulse in the index-two case, suppose that for the given DAE (4.1) we have  $b = \tilde{b} + [b]H_{t_0}$  where  $\tilde{b}$  is the smooth part of the function and  $[b]$  a jump at the time  $t_0$ . Then, we have  $[u_0]H_{t_0} = B(t_0)^T(A(t_0)A(t_0)^T + B(t_0)B(t_0)^T)^{-1}[b]H_{t_0}$  which implies that  $u_0$  has the impulse  $B^T(t_0)(A(t_0)A(t_0)^T + B(t_0)B(t_0)^T)^{-1}[b]\delta_{t_0}$ . Accordingly, the right side  $b_1 = D(b - Au_0 - Bu_0)$  has the impulse

$$(6.8) \quad \beta_1 \delta_{t_0} = -DAB^T(AA^T + BB^T)^{-1}(t_0)[b]\delta_{t_0}.$$

Now let

$$(6.9) \quad A_1 \dot{x}_1 + B_1 x_1 = b_1, \quad b_1 = \tilde{b}_1 + [b_1]H_{t_0} + \beta_1 \delta_{t_0}$$

be the reduced equation and consider its solution in the form  $x_1 = \tilde{x}_1 + [x_1]H_{t_0} + \xi_1 \delta_{t_0}$ . By substituting this into (6.5) and comparing terms we obtain the conditions

$$A_1(t_0)\xi_1 = 0, \quad A_1(t_0)[x_1] + B_1(t_0)\xi_1 = \beta_1, \quad Q_1(t_0)B_1(t_0)[x_1] = Q_1(t_0)[b_1]$$

which can be combined into the two systems

$$(6.10a) \quad (A_1(t_0) + Q_1(t_0)B_1(t_0))\xi_1 = Q_1(t_0)\beta_1$$

$$(6.10b) \quad (A_1(t_0) + Q_1(t_0)B_1(t_0))[x_1] = Q_1(t_0)[b_1] + \beta_1 - B_1(t_0)\xi_1.$$

Since the DAE is assumed to have index two, the matrix  $A_1(t_0) + Q_1(t_0)B_1(t_0)$  is nonsingular and hence the two systems (6.10a/b) can be solved successively. A brief calculation shows that (6.10b) reduces to (4.12) exactly if  $\beta_1 - B_1(t_0)\xi_1 = 0$ .

The relations (6.8), (6.10a/b) were incorporated into LTV2XE to allow for the computation of the jump and impulse at any point  $t_0$  where the right side  $b$  of the original DAE (4.1) has a jump.

As numerical example we consider the following index-two problem

$$(6.11a) \quad \begin{cases} \dot{x}_1 + x_1 - x_2 - x_4 - x_5 = 0 \\ \dot{x}_2 + x_1 - x_2 + tx_3 - x_5 = 0 \\ \dot{x}_3 - tx_1 - x_3 - tx_4 = 0 \\ \dot{x}_4 + (t-1)x_2 + x_3 - tx_4 = 0 \\ t^2x_1 + (1-t)^2x_2 + (t-2)x_3 = \tau(t) \end{cases}$$

where  $\tau(t) = 1$  for  $t \leq 1$  and  $\tau(t) = -1$  for  $t > 1$ . For the consistent starting point

$$(6.11b) \quad x_1 = 0.5, x_2 = 0.0, x_3 = -0.5, x_4 = 0.0, x_5 = 0.0,$$

Table 6.2 shows all steps computed by LTV2XE for the problem (6.11a/b) for  $0 \leq t \leq 1.2$ . A relative tolerance of  $10^{-5}$  and a maximal step of 0.1 was used.

The discontinuity at  $t = 1$  causes a recalculation of the point obtained at  $t = 1$  from which the solution proceeded. Clearly, in order to capture the discontinuity exactly at  $t = 1$ , this value has to be included in the list of required output-points of the code. This was indicated in the table by a dividing line. At any jump point the output of the code includes the values of the jump and the impulse of the solution. In this case, we found that at  $t = 1$  the solution has the jump  $(-2, -2, 0, 0, 2)^T H_1$  and the impulse  $(0, 0, 0, 0, 2)^T \delta_1$ . Of course, the jump is also clearly seen in Table 6.2.

In analogy with the simple problem (6.5) we approximate the step by

$$(6.11) \quad \bar{\tau}(t) = \begin{cases} 1, & \text{for } 0 \leq t < 1 - \epsilon, \\ \frac{1}{2}\sigma(t)^3 - \frac{3}{2}\sigma(t), & \text{for } 1 - \epsilon \leq t \leq 1 + \epsilon, \quad \sigma(t) = \frac{t-1}{\epsilon}, \\ -1, & \text{for } t > 1 + \epsilon, \end{cases}$$

$t$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$
0.0000	0.5000	0.0000	-0.5000	0.0000	0.0000
0.5000(-1)	0.4724	-0.2763(-1)	-0.5250	0.2500(-1)	-0.1340
0.1500	0.3984	-0.1010	-0.5751	0.7502(-1)	-0.4573
0.2500	0.2941	-0.2033	-0.6263	0.1252	-0.8372
0.3475	0.1612	-0.3317	-0.6788	0.1752	-1.184
0.4475	0.3369(-2)	-0.4816	-0.7384	0.2297	-1.351
0.5406	-0.1384	-0.6120	-0.8014	0.2875	-1.193
0.6406	-0.2494	-0.7057	-0.8780	0.3635	-0.7034
0.7060	-0.2862	-0.7284	-0.9317	0.4246	-0.3105
0.7919	-0.2890	-0.7094	-1.003	0.5225	0.1445
0.8639	-0.2570	-0.6571	-1.060	0.6224	0.3976
0.9305	-0.2073	-0.5875	-1.105	0.7306	0.5071
0.9969	-0.1452	-0.5060	-1.141	0.8545	0.5046
1.000	-0.1421	-0.5020	-1.142	0.8608	0.5020
1.000	-2.142	-2.502	-1.142	0.8608	2.502
1.002	-2.137	-2.496	-1.146	0.8644	2.504
1.003	-2.134	-2.493	-1.149	0.8663	2.505
1.010	-2.111	-2.468	-1.167	0.8818	2.511
1.034	-2.037	-2.387	-1.224	0.9341	2.513
1.058	-1.962	-2.303	-1.281	0.9910	2.490
1.082	-1.887	-2.220	-1.334	1.052	2.444
1.105	-1.813	-2.136	-1.386	1.116	2.379
1.129	-1.740	-2.053	-1.435	1.184	2.296
1.151	-1.668	-1.971	-1.481	1.255	2.197
1.174	-1.597	-1.889	-1.523	1.330	2.086
1.196	-1.528	-1.810	-1.563	1.408	1.963
1.200	-1.516	-1.797	-1.569	1.422	1.941

Table 6.2: Solution of (6.11a/b)

with small  $\epsilon > 0$ , then we expect the solution to approximate the impulse  $(0, 0, 0, 0, 2)^T \delta_1$ .

Figure 6.1 show the fifth components in the case of  $\epsilon = 0.05$ . For smaller values of  $\epsilon$  the system becomes too stiff to capture the impulse.



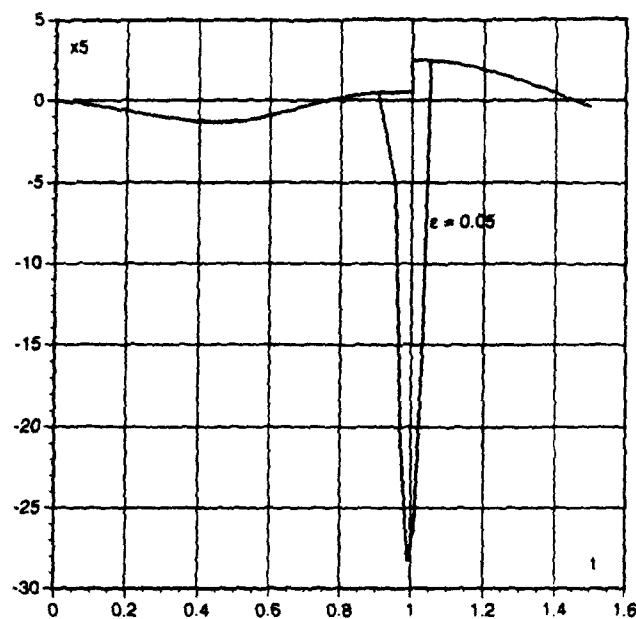


FIGURE 6.1

It may be noted that we did not succeed to compute the solutions of this problem either with DASSL (see e.g.[BCP89]) or RADAU5 (see [HW91]). Both codes failed at start-up.

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